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On the instability of solutions of certain non-autonomous vector differential equations of fifth order

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Abstract. By using Lyapunov's function approach [13], some new results were established, which guarantee that the zero solution of non-linear vector differential equations of the form

$$X^{(5)} + a(t)\Psi(\dot{X},\ddot{X})\dddot{X} + b(t)\Phi(X,\dot{X},\ddot{X},\dddot{X},X^{(4)}) + c(t)\Theta(\dot{X}) + F(X) = 0$$

is unstable.

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§1. Introduction

Instability problems for various linear and non-linear differential equations of higher order, third-, fourth-, fifth-, sixth-, seventh and eighth orders, have been studied by many authors. For some related contributors, we refer to the papers of Ezeilo ([1, 2, 3, 4, 5], Krasovskii [6], Liao and Lu [8], Li and Yu [9], Li and Duan [10], Lu [11], Lu and Liao [12], Sadek ([15, 16]), Skrapek ([17, 18]), Sun and Hou [19], Tejumola [20], Tiryaki ([21, 22, 23]), Tunç ([24, 25, 26, 27, 28, 29, 30, 31, 32]), Tunç and Sevli [33], C.Tunç and E. Tunç ([34, 35, 36, 37]), E. Tunç [38] and the references cited therein. In all of the above mentioned works, taking into consideration Krasovskii's criteria [6] and using the Lyapunov's second (or direct) method [13] the results there were proved by the authors. The reason for this case is, perhaps, due to the effectiveness of Lyapunov's second method [13] and Krasovskii's criterion [6]. Now, it should be better to summarize some works, in particular, focused on the instability of nonlinear differential equations of fifth order. Namely, according to our observations in the literature, first, in 1978 and 1979 for the case $n = 1$, Ezeilo ([2, 3, 4]) investigated the instability of zero solution for the following nonlinear scalar differential equations:

$$x^{(5)} + a_1x^{(4)} + a_2\dddot{x} + a_3\ddot{x} + a_4\dot{x} + f(x) = 0,$$
$$x^{(5)} + a_1x^{(4)} + a_2\dddot{x} + h(\dot{x})\ddot{x} + g(x)\dot{x} + f(x) = 0,$$
$$x^{(5)} + \psi(\ddot{x})\dddot{x} + \phi(\ddot{x}) + \theta(\dot{x}) + f(x) = 0$$

and

$$x^{(5)} + a_1x^{(4)} + a_2\dddot{x} + g(\dot{x})\ddot{x} + h(x,\dot{x},\ddot{x},\dddot{x},x^{(4)})\dot{x} + f(x) = 0,$$

respectively. Later, in 1987, Tiryaki [23] established a result on the instability of zero solution of scalar differential equation

$$x^{(5)} + a_1x^{(4)} + k(x,\dot{x},\ddot{x},\dddot{x},x^{(4)})\dddot{x} + g(\dot{x})\ddot{x} + h(x,\dot{x},\ddot{x},\dddot{x},x^{(4)})\dot{x} + f(x) = 0.$$

On the other hand, in 2003, Sadek [16] studied the instability behaviors of solutions of fifth order nonlinear vector differential equations described as follows

$$X^{(5)} + \Psi(\ddot{X})\dddot{X} + \Phi(\ddot{X}) + \Theta(\dot{X}) + F(X) = 0$$

and

$$X^{(5)} + AX^{(4)} + B\dddot{X} + H(\dot{X})\ddot{X} + G(X)\dot{X} + F(X) = 0.$$

More recently, Tunç ([25, 28]) and Tunç and Sevli [33], respectively, also gave sufficient conditions which guarantee that the zero solution of the vector differential equations of the form

$$\begin{aligned} &X^{(5)} + AX^{(4)} + \Psi(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\dddot{X} + G(\dot{X})\ddot{X} \\ &\quad + H(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\dot{X} + F(X) = 0, \\ &X^{(5)} + AX^{(4)} + B(t)\Psi(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\dddot{X} + C(t)G(\dot{X})\ddot{X} \\ &\quad + D(t)H(X,\dot{X},\ddot{X},\dddot{X},X^{(4)})\dot{X} + E(t)F(X) = 0 \end{aligned}$$

and

$$X^{(5)} + \Psi(\dot{X},\ddot{X})\dddot{X} + \Phi(X,\dot{X},\ddot{X}) + \Theta(\dot{X}) + F(X) = 0$$

is unstable.

This paper is concerned with the instability of the zero solution of fifth-order nonlinear vector differential equation described by

$$X^{(5)} + a(t)\Psi(\dot{X},\ddot{X})\dddot{X} + b(t)\Phi(X,\dot{X},\ddot{X},\dddot{X},X^{(4)}) + c(t)\Theta(\dot{X}) + F(X) = 0 \tag{1.1}$$

in the real Euclidean space $\mathbb{R}^n$ (with the usual norm denoted in what follows by $\|.\|$), where $X \in \mathbb{R}^n$; Ψ is an $n \times n$-symmetric continuous matrix function depending, in each case, on the arguments shown; $a : \mathbb{R}^+ \to \mathbb{R}^+$, $b : \mathbb{R}^+ \to \mathbb{R}^+$, $c : \mathbb{R}^+ \to \mathbb{R}^+$, $\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\Theta : \mathbb{R}^n \to \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ and $\Phi(X, \dot{X}, 0, \dddot{X}, X^{(4)}) = \Theta(0) = F(0)=0$. It is also supposed that the functions a, b, c, Φ, Θ and F are continuous for arguments shown explicitly. Throughout this paper, we consider, instead of equation (1.1), the equivalent differential system:

$$\dot{X} = Y, \quad \dot{Y} = Z, \quad \dot{Z} = W, \quad \dot{W} = U,$$

$$\dot{U} = -a(t)\Psi(Y, Z)W - b(t)\Phi(X, Y, Z, W, U) - c(t)\Theta(Y) - F(X), \tag{1.2}$$

which was obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$, $\dddot{X} = W$, $X^{(4)} = U$ from equation (1.1). For the sake of the brevity, we assume that the symbols $J\,(\Psi(Y, Z)Z\,|Y\,)$, $J\,(\Psi(Y, Z)\,|Z\,)$, $J_\Theta(Y)$ and $J_F(X)$, respectively, denote the Jacobian matrices as follows:

$$J\,(\Psi(Y, Z)Z\,|Y\,) = \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n}\psi_{ik}z_k\right) = \left(\sum_{k=1}^{n}\frac{\partial\psi_{ik}}{\partial y_j}z_k\right),$$

$$J\,(\Psi(Y, Z)\,|Z\,) = \left(\frac{\partial}{\partial z_j}\sum_{k=1}^{n}\psi_{ik}\right) = \left(\sum_{k=1}^{n}\frac{\partial\psi_{ik}}{\partial z_j}\right),$$

$$J_\Theta(Y) = \left(\frac{\partial\theta_i}{\partial y_j}\right), \qquad J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right), \qquad (i, j = 1,\, 2,\, \ldots,\, n),$$

where $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_n)$, (ψ_{ik}), $(i, k = 1, 2, \ldots, n)$, $(\theta_1, \theta_2, \ldots, \theta_n)$ and $(f_1, f_2, \cdots, f_n)$ are the components of X, Y, Z, Ψ, Θ and F, respectively. In addition to these, it is assumed, as basic throughout the paper, that the Jacobian matrices $J\,(\Psi(Y, Z)Z\,|Y\,)$, $J\,(\Psi(Y, Z)\,|Z\,)$, $J_\Theta(Y)$ and $J_F(X)$ exist and are continuous and symmetric. The symbol $\langle X, Y\rangle$ corresponding to any pair$X$, Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$, and $\lambda_i(A)$, $(A = (a_{ij}),\ (i, j = 1, 2, \ldots, n))$, are the eigenvalues of the $n \times n$-symmetric matrix A and the matrix $A = (a_{ij})$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \mathbb{R}^n$ and X^T denotes the transpose of X.

Finally, the motivation for the present work has been inspired basically by the papers just mentioned above.

§2. Preliminaries

In order to reach our main results, we state a basic theorem for the general non-autonomous differential system and also express a well-known lemma which plays an essential role throughout the proofs of main results of this paper. Now, consider the differential system

$$\dot{x} = f(t,x), \quad x(t_0) = x_0, \quad t \geq 0, \tag{2.1}$$

where $f \in C\left[\mathbb{R}^+ \times S_{(\rho)}, \mathbb{R}^n\right]$ and $S_{(\rho)} = [x \in \mathbb{R}^n : |x| < \rho]$. We assume, for convenience, that the solutions $x(t) = x(t,t_0,x_0)$ of (2.1) exist, and are unique for $t \geq t_0$ and $f(t,0) = 0$ so that we have trivial solution $x = 0$.

First, we state that the following fundamental instability theorem.

Theorem 2.1. *Assume that there exists a $t_0 \in \mathbb{R}^+$ and an open set $U \subset S_{(\rho)}$ such that $V \in C^1\left[[t_0,\infty) \times S_{(\rho)}, \mathbb{R}^+\right]$ for (t,x) from $[t_0,\infty) \times U$,*

(i) $0 < V(t,x) \leq a(|x|)$, *$a \in \kappa$;*

(ii) *either $V'(t,x) \geq b(|x|)$, $b \in \kappa$, $\kappa = [\sigma \in C\left[\ [t_0,\rho)\,,\ \mathbb{R}^+\right]\ ]$ such that $\sigma(t)$ is strictly increasing and $\sigma(0) = 0$ or $V'(t,x) = CV(t,x) + \omega(t,x)$, where $C > 0$ and $\omega \in C\left[[t_0,\infty) \times U, \mathbb{R}^+\right]$;*

(iii) *$V(t,x) = 0$ on $[t_0,\infty) \times \left(\partial U \cap S_{(\rho)}\right)$, ∂U denotes boundary of U and $0 \in \partial U$.*

Then the trivial solution $x = 0$ of system (2.1) is unstable.

Proof. See Lakshmikantham et al. [7, Theorem 1.1.9]. □

Lemma 2.2. *Let A be a real symmetric $n \times n$-matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \quad (i = 1,\, 2,\, \ldots,\, n),$$

where a', a are constants.

Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle\,.$$

Proof. See Mirsky [14]. □

§3. Main Results

In this section we establish some sufficient conditions which guarantee that zero solution of equation (1.1) is unstable.

Theorem 3.1. *In addition to the basic assumptions imposed on a, b, c, Ψ, Φ, Θ and F that appeared in equation (1.1), we assume the following conditions are satisfied:*

(i) $a_0 \geq a(t) \geq 1$, $b_0 \geq b(t) \geq 1$, $c_0 \geq c(t) \geq 1$, $a'(t) < 0$ *and* $c'(t) \leq 0$ *for all* $t \in \mathbb{R}^+$,
where a_0, b_0 *and* c_0 *are some positive constants.*

(ii) $F(0) = 0$ *and* $F(X) \neq 0$ *if* $X \neq 0$, *and the Jacobian matrices* $J_F(X)$, $J_\Theta(Y)$ *are symmetric and* $-f_0 \leq \lambda_i(J_F(X)) < 0$, $\quad 0 < \lambda_i(J_\Theta(Y)) \leq \theta_0$, $\quad (i = 1, 2, \ldots, n)$, *for all* X, $Y \in \mathbb{R}^n$,
where f_0 *and* θ_0 *are some positive constants.*

(iii) $\Phi(X, Y, 0, W, U) = 0$, $\Phi(X, Y, Z, W, U) \neq 0$ *if* $Z \neq 0$,
and $\sum_{i=1}^{n} z_i \phi_i(X, Y, Z, W, U) \geq 0$ *for all* X, Y, Z, W, $U \in \mathbb{R}^n$,
where $\Phi(X, Y, Z, W, U) = (\phi_1(X, Y, Z, W, U), \ldots, \phi_n(X, Y, Z, W, U))$,

(iv) *The matrices* $\Psi(Y, Z)$*and* $J\left(\Psi(Y, Z)Z \,|Y\right)$ *are symmetric;*
$0 < \lambda_i(\Psi(Y, Z)) \leq \psi_0$ *and* $J\left(\Psi(Y, Z)Z \,|Y\right) \leq 0$ *for all* Y, $Z \in \mathbb{R}^n$, *where* ψ_0 *is a positive constant.*

Then the zero solution $X = 0$ *of equation* (1.1) *is unstable.*

Proof. For the proof of Theorem 3.1, we define the Lyapunov function $V_0 = V_0(t, X, Y, Z, W, U)$ as follows:

$$V_0 = \frac{1}{2}\langle W, W\rangle - \langle Z, U\rangle - \langle Y, W\rangle - c(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma - a(t)\int_0^1 \langle \sigma\Psi(Y, \sigma Z)Z, Z\rangle\, d\sigma \tag{3.1}$$

Taking notice of (3.1), we see that $V_0(t, 0, 0, 0, 0, 0) = 0$. Next, evidently, ones can easily get that

$$V_0(t, 0, 0, 0, \varepsilon, 0) = \frac{1}{2}\langle \varepsilon, \varepsilon\rangle = \frac{1}{2}\|\varepsilon\|^2 > 0$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \mathbb{R}^n$. In view of the function $V_0 = V_0(t, X, Y, Z, W, U)$, the assumptions of Theorem 3.1, the properties of symmetric matrices, the above Lemma and Cauchy-Schwarz inequality $|\langle X, Y\rangle| \leq \|X\|\,\|Y\|$, one can easily conclude from (3.1) that there is a positive constant K_1 such that

$$V_0(t, X, Y, Z, W, U) \leq K_1\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2\right).$$

Now, consider $(X, Y, Z, W, U) = (X(t), Y(t), Z(t), W(t), U(t))$ as an arbitrary solution of the system (1.2). Differentiating the Lyapunov function in (3.1) and making use of the system (1.2), we have that

$$\dot{V}_0 = \frac{d}{dt}V_0(t,X,Y,Z,W,U) = \langle Z, b(t)\Phi(X,Y,Z,W,U)\rangle - \langle Y, J_F(X)Y\rangle$$
$$+ \langle a(t)\Psi(Y,Z)W, Z\rangle + \langle c(t)\Theta(Y), Z\rangle$$
$$- \frac{d}{dt}c(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle \, d\sigma - \frac{d}{dt}a(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle \, d\sigma. \tag{3.2}$$

Recall that

$$\frac{d}{dt}a(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle \, d\sigma = a(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, W\rangle \, d\sigma$$
$$+ a(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)W, Z\rangle \, d\sigma + a(t)\int_0^1 \left\langle \sigma^2 J(\Psi(Y,\sigma Z)\,|Z)WZ\ , Z\right\rangle d\sigma$$
$$+ a(t)\int_0^1 \langle \sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ , Z\rangle \, d\sigma + a'(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle \, d\sigma$$
$$= a(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)W, Z\rangle \, d\sigma + a(t)\int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma\Psi(Y,\sigma Z)W, Z\rangle \, d\sigma$$
$$+ a(t)\int_0^1 \langle \sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ , Z\rangle \, d\sigma + a'(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle \, d\sigma$$
$$= \sigma^2 \langle a(t)\Psi(Y,\sigma Z)W, Z)\rangle \,\Big|_0^1 + a(t)\int_0^1 \langle \sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ , Z\rangle \, d\sigma$$
$$+ a'(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle \, d\sigma$$
$$= \langle a(t)\Psi(Y,Z)W, Z\rangle + a(t)\int_0^1 \langle \sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ , Z\rangle \, d\sigma$$
$$+a'(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle \, d\sigma. \tag{3.3}$$

Similarly, it is clear that

$$\begin{aligned}
&\frac{d}{dt}c(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma \\
&= c(t)\int_0^1 \sigma\, \langle J_\Theta(\sigma Y)Z, Y\rangle d\sigma \\
&\quad + c(t)\int_0^1 \langle \Theta(\sigma Y), Z\rangle\, d\sigma + c'(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma \\
&= c(t)\int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \Theta(\sigma Y), Z\rangle\, d\sigma + c(t)\int_0^1 \langle \Theta(\sigma Y), Z\rangle\, d\sigma \\
&\quad + c'(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma \\
&= \sigma\, \langle c(t)\Theta(\sigma Y), Z\rangle\, \big|_0^1 + c'(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma \\
&= \langle c(t)\Theta(Y), Z\rangle + c'(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma. \qquad (3.4)
\end{aligned}$$

Now, since

$$\Theta(0) = 0 \text{ and } \frac{\partial}{\partial\sigma}\Theta(\sigma Y) = J_\Theta(\sigma Y)Y,$$

we can write

$$\Theta(Y) = \int_0^1 J_\Theta(\sigma Y)Y\, d\sigma. \qquad (3.5)$$

Hence, the expression (3.5) leads that

$$c'(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma = c'(t)\int_0^1\int_0^1 \langle \sigma_1 J_\Theta(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_2 d\sigma_1. \qquad (3.6)$$

Thus, in view of (3.4) and (3.6), it follows that

$$\begin{aligned}
\frac{d}{dt}c(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma &= \langle c(t)\Theta(Y), Z\rangle \\
&\quad + c'(t)\int_0^1\int_0^1 \langle \sigma_1 J_\Theta(\sigma_1\sigma_2 Y)Y, Y\rangle d\sigma_2 d\sigma_1. \qquad (3.7)
\end{aligned}$$

Now, evidently, the expressions (3.2), (3.3) and (3.7) and the assumptions of Theorem 3.1, together, yield that

$$\begin{aligned}
\dot{V}_0 &= \langle Z, b(t)\Phi(X,Y,Z,W,U)\rangle - \langle Y, J_F(X)Y\rangle \\
&\quad - a(t)\int_0^1 \langle \sigma J(\Psi(Y,\sigma Z)Z\,|Y)Z\ , Z\rangle\, d\sigma \\
&\quad - a'(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle\, d\sigma - c'(t)\int_0^1\int_0^1 \langle \sigma_1 J_\Theta(\sigma_1\sigma_2 Y)Y, Y\rangle\, d\sigma_2 d\sigma_1 \\
&\geq \langle Z, \Phi(X,Y,Z,W,U)\rangle - \langle Y, J_F(X)Y\rangle - a'(t)\int_0^1 \langle \sigma\Psi(Y,\sigma Z)Z, Z\rangle\, d\sigma \\
&\quad - c'(t)\int_0^1\int_0^1 \langle \sigma_1 J_\Theta(\sigma_1\sigma_2 Y)Y, Y\rangle\, d\sigma_2 d\sigma_1 > 0.
\end{aligned}$$

So, the assumptions of Theorem 3.1 imply that $\dot{V}_0(t) \geq K_2\left(\|Y\|^2 + \|Z\|^2\right)$ > 0 for all $t \geq 0$, where K_2 is a positive constant, say infinite inferior limit of the function $\dot{V}_0$. Additionally, $\dot{V}_0 = 0$ $(t \geq 0)$ necessarily implies that $Y = 0$ for all $t \geq 0$, and hence also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$, $W = \ddot{Y} = 0$, $U = \dddot{Y} = 0$ for all $t \geq 0$. By using the expressions

$$X = \xi, \quad Y = Z = W = U = 0$$

in the system (1.2), it can be seen easily that $F(\xi) = 0$ which necessarily leads that $\xi = 0$ because $F(0) = 0$. In view of the above discussion, clearly, it follows that

$$X = Y = Z = W = U = 0 \text{ for all } t \geq 0.$$

Therefore, subject to the assumptions of Theorem 3.1, the function V_0 has the entire the criteria of the theorem of Lakshmikantham et al. [7, Theorem 1.1.9]. Thus, the basic properties of the function $V_0(t,X,Y,Z,W,U)$, which were proved above verify that the zero solution of system (1.2) is unstable. The system of equations (1.2) is equivalent to differential equation (1.1) and hence the proof of Theorem 3.1 is now complete. □

Our last result is the following theorem.

Theorem 3.2. *In addition to the basic assumptions imposed on a, b, c, Ψ, Φ, Θ and F that appeared in equation* (1.1), *we assume the following conditions are satisfied:*

(i) $a_0 \geq a(t) \geq 1$, $b_0 \geq b(t) \geq 1$, $-c_0 \leq c(t) \leq -1$, $a'(t) > 0$ *and* $c'(t) \leq 0$ *for all* $t \in \mathbb{R}^+$, *where* a_0, b_0 *and* c_0 *are some positive constants.*

(ii) *$F(0) = 0$ and $F(X) \neq 0$ if $X \neq 0$, and the Jacobian matrices $J_\Theta(Y)$, $J_F(X)$ are symmetric and $f_0 \geq \lambda_i(J_F(X)) > 0$ and $-\theta_0 \leq \lambda_i(J_\Theta(Y)) < 0$ for all X, $Y \in \mathbb{R}^n$, where f_0 and θ_0 are some positive constants.*

(iii) *$\Phi(X, Y, 0, W, U) = 0$, $\Phi(X, Y, Z, W, U) \neq 0$ if $Z \neq 0$, and $\sum_{i=1}^{n} z_i\phi_i(X, Y, Z, W, U) \leq 0$ for all X, Y,Z, W, $U \in \mathbb{R}^n$, where $\Phi(X, Y, Z, W, U) = (\phi_1(X, Y, Z, W, U), \ldots, \phi_n(X, Y, Z, W, U))$.*

(iv) *The matrices $\Psi(Y, Z)$ and $J\left(\Psi(Y, Z)Z\,|Y\right)$ are symmetric; $0 < \lambda_i(\Psi(Y, Z)) \leq \psi_0$ and $J\left(\Psi(Y, Z)Z\,|Y\right) \geq 0$ for all Y, $Z \in \mathbb{R}^n$, where ψ_0 is a positive constant.*

Then the zero solution $X = 0$ of equation (1.1) *is unstable.*

Proof. As similar in the proof of Theorem 3.1, we now define for the proof of Theorem 3.2 the Lyapunov function $V_1 = V_1(t, X, Y, Z, W, U)$ such that $V_1 = -V_0$, where V_0 is defined as the same as in (3.1), that is,

$$V_1 = \frac{1}{2}\langle W, W\rangle + \langle Z, U\rangle + \langle Y, W\rangle + c(t)\int_0^1 \langle \Theta(\sigma Y), Y\rangle\, d\sigma + a(t)\int_0^1 \langle \sigma\Psi(Y, \sigma Z)Z, Z\rangle\, d\sigma.$$

Clearly, $V_1(t, 0, 0, 0, 0, 0) = 0$ and in view of conditions (i) and (iv) of Theorem 3.2, we have that

$$\begin{aligned} V_1(t, 0, 0, \varepsilon, 0, \varepsilon) &= \langle \varepsilon, \varepsilon\rangle + a(t)\int_0^1 \langle \sigma\Psi(0, \sigma\varepsilon)\varepsilon, \varepsilon\rangle\, d\sigma \\ &\geq \|\varepsilon\|^2 + \int_0^1 \langle \sigma\Psi(0, \sigma\varepsilon)\varepsilon, \varepsilon\rangle\, d\sigma > 0 \end{aligned}$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \mathbb{R}^n$. The rest of the proof is similar to that of Theorem3.1, except for some minor modifications, hence it is omitted. □

Remark 3.3. It should be noted that, for the case $n = 1$, the result of Ezeilo [2; Theorem3.2] is a special case of our first result. Next, the results constituted here give an additional result to that of established by Sadek [16; Theorem3.2] and Tunç and Sevli [33].

Example: As a special case of the system (1.2), let us choose, for $n = 5$, Ψ, Φ, Θ and F as:

$$\Psi(Z) = \begin{bmatrix} 1+\frac{1}{1+z_1^2} & 0 & 0 & 0 & 0 \\ 0 & 1+\frac{1}{1+z_2^2} & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{1}{1+z_3^2} & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{1+z_4^2} & 0 \\ 0 & 0 & 0 & 0 & 1+\frac{1}{1+z_5^2} \end{bmatrix},$$

$$\Phi(Z) = \begin{bmatrix} z_1^3 + z_1^5 \\ z_2^3 + z_2^5 \\ z_3^3 + z_3^5 \\ z_4^3 + z_4^5 \\ z_5^3 + z_5^5 \end{bmatrix}, \quad \Theta(Y) = \begin{bmatrix} y_1 + \arctan y_1 \\ y_2 + \arctan y_2 \\ y_3 + \arctan y_3 \\ y_4 + \arctan y_4 \\ y_5 + \arctan y_5 \end{bmatrix}$$

and

$$F(X) = \begin{bmatrix} -x_1 - \arctan x_1 \\ -x_2 - \arctan x_2 \\ -x_3 - \arctan x_3 \\ -x_4 - \arctan x_4 \\ -x_5 - \arctan x_5 \end{bmatrix}.$$

Then, respectively, we get

$$\lambda_1(\Psi(Z)) = 1 + \frac{1}{1+z_1^2}, \quad \lambda_2(\Psi(Z)) = 1 + \frac{1}{1+z_2^2}, \quad \lambda_3(\Psi(Z)) = 1 + \frac{1}{1+z_3^2},$$
$$\lambda_4(\Psi(Z)) = 1 + \frac{1}{1+z_4^2}, \quad \lambda_5(\Psi(Z)) = 1 + \frac{1}{1+z_5^2},$$

$$J_\Theta(Y) = \begin{bmatrix} 1+\frac{1}{1+y_1^2} & 0 & 0 & 0 & 0 \\ 0 & 1+\frac{1}{1+y_2^2} & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{1}{1+y_3^2} & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{1+y_4^2} & 0 \\ 0 & 0 & 0 & 0 & 1+\frac{1}{1+y_5^2} \end{bmatrix},$$

$$\lambda_1(J_\Theta(Y)) = 1 + \frac{1}{1+y_1^2}, \quad \lambda_2(J_\Theta(Y)) = 1 + \frac{1}{1+y_2^2},$$
$$\lambda_3(J_\Theta(Y)) = 1 + \frac{1}{1+y_3^2}, \quad \lambda_4(J_\Theta(Y)) = 1 + \frac{1}{1+y_4^2},$$
$$\lambda_5(J_\Theta(Y)) = 1 + \frac{1}{1+y_5^2},$$

$$J_F(X) = \begin{bmatrix} -1-\frac{1}{1+x_1^2} & 0 & 0 & 0 & 0 \\ 0 & -1-\frac{1}{1+x_2^2} & 0 & 0 & 0 \\ 0 & 0 & -1-\frac{1}{1+x_3^2} & 0 & 0 \\ 0 & 0 & 0 & -1-\frac{1}{1+x_4^2} & 0 \\ 0 & 0 & 0 & 0 & -1-\frac{1}{1+x_5^2} \end{bmatrix}$$

and

$$\lambda_1(J_F(X)) = -1 - \frac{1}{1+x_1^2}, \quad \lambda_2(J_F(X)) = -1 - \frac{1}{1+x_2^2},$$
$$\lambda_3(J_F(X)) = -1 - \frac{1}{1+x_3^2}, \quad \lambda_4(J_F(X)) = -1 - \frac{1}{1+x_4^2},$$
$$\lambda_5(J_F(X)) = -1 - \frac{1}{1+x_5^2}.$$

Hence,

$1 \le \lambda_i(\Psi(Z)) \le 2$ for all z_1, z_2, z_3, z_4 and z_5;

$1 \le \lambda_i(J_\Theta(Y)) \le 2$ for all y_1, y_2, y_3, y_4 and y_5;

$-2 \le \lambda_i(J_F(X)) \le -1$ for all x_1, x_2, x_3, x_4, x_5, $(i = 1, 2, 3, 4, 5)$, and

$$\sum_{i=1}^{3} z_i \Phi_i(Z) = z_1^4 + z_1^6 + z_2^4 + z_2^6 + z_3^4 + z_3^6 + z_4^4 + z_4^6 + z_5^4 + z_5^6 \ge 0$$

for all z_1, z_2, z_3, z_4 and z_5.

Remark 3.4. Thus, for the special case $a(t) = b(t) = c(t) = 1$, if the assumptions of Theorem3.1 and Theorem3.2 hold, then the Lyapunov functions V_0 and V_1 have the entire criteria of Krasovskii's [6]. For instance, when $a(t) = b(t) = c(t) = 1$, the Lyapunov function continuous function $V_0 = V_0(X, Y, Z, W, U)$ satisfies the following Krasovskii properties:

(K_1) In every neighborhood of $(0, 0, 0, 0, 0)$ there exists a point $(\xi, \eta, \zeta, \mu, \rho)$ such that $V_0(\xi, \eta, \zeta, \mu, \rho) > 0$;

(K_2) the time derivative $\dot{V}_0 = \frac{d}{dt}V_0(X, Y, Z, W, U)$ along solution paths of the system (1.2) is positive semi-definite; and

(K_3) the only solution $(X, Y, Z, W, U) = (X(t), Y(t), Z(t), W(t), U(t))$ of the system (1.2) which satisfies $\dot{V}_0 = 0 (t \ge 0)$ is the trivial solution $(0, 0, 0, 0, 0)$.

Hence, this properties show that the zero solution of equation (1.1) is unstable.

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